# Shortest Total Distance Traveled Among Curves, Surfaces and Lagrange Multipliers

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**Abstract**

*This paper was inspired by students at Guangzhou University in 2014, the problems are variations from those described in [1]. We seek the shortest total distance from curve 1 to curve 2, curve 2 to curve 3 and curve 3 back to curve 1. In 3D, we want to find the shortest total distance between surface 1 and surface 2, surface 2 and surface 3, surface 3 and surface 4, and finally surface 4 and surface 1. We assume curves and surfaces are not intersecting. We start with the simplest case for circles in 2D and we link to higher dimensions through the Lagrange multipliers method.*

## 1 Introduction

In *[1]*, we considered three given disjoint curves, $C_1, C_2$ and $C_3$ respectively in the plane, and we found the minimum total (squared) distance from $C_1$ to $C_2$ and $C_1$ to $C_3$. We also worked on four non-intersecting **convex** surfaces $S_1, S_2, S_3$ and $S_4$ in $\mathbb{R}^3$ and we found the the minimum total (squared) distance from $S_1$ to $S_2$, $S_1$ to $S_3$, and $S_1$ to $S_4$. In this paper, we discuss the following problems which were raised during the first author's research visit at Guangzhou University during February and April of 2014. Given three disjoint curves, $C_1, C_2$ and $C_3$ respectively in $\mathbb{R}^2$, we need to find the minimum total distance from $C_1$ to $C_2$, $C_2$ to $C_3$ and $C_3$ to $C_1$. Next, we consider four disjoint **convex** surfaces $S_1, S_2, S_3$ and $S_4$ and we shall find the minimum total distance from $S_1$ to $S_2$, $S_2$ to $S_3$, $S_3$ to $S_4$ and $S_4$ to $S_1$. We note that the total squared distance is obviously different from the exact total distance. However, in this paper, we do not distinguish the differences between these two when we are proving theorems. We will only use true total distance, $\|\mathbf{x} - \mathbf{y}\|$, for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ in actual computations with Maple. The method of finding the extremum of the total (squared) distance is an application of the Lagrange Multipliers Method, which we summarize in Corollary. Theorem 6 is a generalization of Corollary. Throughout this paper, the curves and surfaces are not intersecting.

## 2 Three circles in 2D

We start with the simple three circles case in 2D, which can be accessible to high school students. We call the three given circles $\Gamma_1, \Gamma_2, \Gamma_3$, with centers $O_1, O_2, O_3$ respectively. It is assumed that no two circles intersect, and no circle is inside another. We want to find three points $A \in \Gamma_1, B \in \Gamma_2$ and $C \in \Gamma_3$ such that the sum $AB + BC + CA$ attains a minimum. We note that $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are closed sets on the Cartesian coordinate system, so the distance $AB + BC + CA$ is a continuous function of $A, B$ and $C$, so distance has a minimum value and such points $A, B$ and $C$ exist.

**Theorem 1** *If $A \in \Gamma_1, B \in \Gamma_2$ and $C \in \Gamma_3$ are such that the sum $AB + BC + CA$ reaches the minimum, and we draw the line $PQ$ passing through $A$ and tangent to $\Gamma_1$ (see Figure 1), we must have $\measuredangle BAP = \measuredangle CAQ$.*

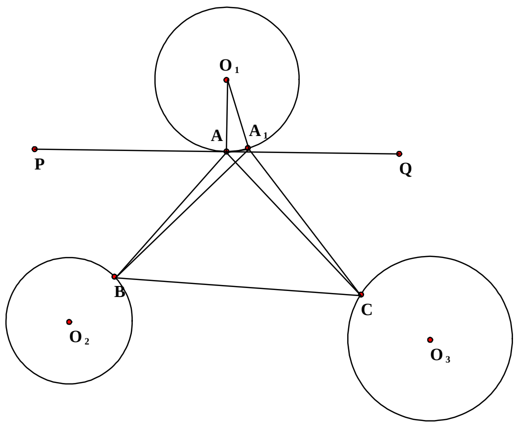


*Figure 1. Tangency and equal angles*

**Proof:** Suppose that $AB = c, AC = b, O_1A = r, \measuredangle BAP = \alpha$ and $\measuredangle CAQ = \beta$. Let $A_1$ be a point on the circle $\Gamma_1$ which is close to $A$, and let $\measuredangle AO_1A_1 = 2\epsilon$. (Here if $A_1$ is inside the angle $\measuredangle O_1AQ$, then $\epsilon > 0$. If $A_1$ is inside the angle $\measuredangle O_1AP$, then $\epsilon < 0$. Now let us calculate the value of $A_1B + A_1C$. We only consider the case in Figure 1 when $\epsilon > 0$. The other case is similar.

First, we note that $AA_1 = 2r \sin\epsilon$, $\measuredangle O_1AA_1 = \frac{\pi}{2} - \epsilon$, $\measuredangle O_1AP = \frac{\pi}{2}$ and since $\measuredangle BAP = \alpha$, we see $\measuredangle BAA_1 = \pi - \alpha + \epsilon$. It follows from the law of cosine that

$$\begin{aligned} BA_1 &= \sqrt{BA^2 + AA_1^2 - 2BA \cdot AA_1 \cos(\pi - \alpha + \epsilon)} \\ &= \sqrt{c^2 + (2r\sin\epsilon)^2 + 4cr\sin\epsilon \cdot \cos(\alpha - \epsilon)} \\ &= \sqrt{c^2 + 4cr\sin\epsilon\cos\alpha\cos\epsilon + 4cr\sin^2\epsilon\cos\alpha + O(\epsilon^2)} \\ &= \sqrt{c^2 + 2cr\sin 2\epsilon\cos\alpha + O(\epsilon^2)} \\ &\approx \sqrt{c^2 + 2cr \cdot 2\epsilon\cos\alpha + O(\epsilon^2)O(\epsilon^2)} \text{ (when } \epsilon \text{ is small)} \\ &= c + 2r\epsilon \cdot \cos\alpha + O(\epsilon^2). \end{aligned} \tag{1}$$

Similarly, we observe that $\measuredangle CAA_1 = \beta + \epsilon$ and it follows from the law of cosine we have

$$\begin{aligned} CA_1 &= \sqrt{CA^2 + AA_1^2 - 2CA \cdot AA_1 \cos(\beta + \epsilon)} \\ &= \sqrt{b^2 + (2r \sin \epsilon)^2 - 4br \sin \epsilon \cdot \cos(\beta + \epsilon)} \\ &= \sqrt{b^2 - 4br \sin \epsilon \cos \beta \cos \epsilon + 4br \sin^2 \epsilon \cos \beta + O(\epsilon^2)} \\ &= \sqrt{b^2 - 2br \sin 2\epsilon \cos \beta + O(\epsilon^2)} \\ &\approx \sqrt{b^2 - 2br \cdot 2\epsilon \cos \beta + O(\epsilon^2)} (\text{when } \epsilon \text{ is small}) \\ &= b - 2r\epsilon \cos \beta + O(\epsilon^2). \end{aligned} \tag{2}$$

Therefore, $CA_1 + BA_1 = b + c + 2r(\cos\alpha - \cos\beta)\epsilon + O(\epsilon^2)$ is a function of $\epsilon$. It is clear that if $AB + BC + CA$ reaches the minimum, then $CA_1 + BA_1 \geq b + c$ must hold, which implies $2r(\cos\alpha - \cos\beta) = 0$ and hence $\alpha = \beta$.

**Theorem 2** *If $A \in \Gamma_1, B \in \Gamma_2$ and $C \in \Gamma_3$ such that the sum $AB + BC + CA$ reaches the minimum, then the three lines $O_1A, O_2B$ and $O_3C$ meet at the incenter $I$ of $\triangle ABC$.*

**Proof:** It follows from theorem 1 that the line $O_1A$ is the angle bisector of $\measuredangle BAC$. Similarly, $O_2B$ and $O_3C$ are the angle bisectors of $\measuredangle CBA$ and $\measuredangle ACB$ respectively. Therefore, three lines meet at the incenter $I$ of $\triangle ABC$.

**Remark:** We note from Theorem 2 that it suggests a way to obtain the required points for achieving the minimum of $AB + BC + CA$ by using a dynamic geometry software, we use *[2]* for demonstration. We choose a moving point $I$, and let $O_1I, O_2I$ and $O_3I$ meet $\Gamma_1, \Gamma_2$ and $\Gamma_3$ at $A, B$ and $C$, respectively. Then we draw lines $AB, BC$ and $CA$, and measure $\measuredangle BAI, \measuredangle CAI, \measuredangle ABI$ and $\measuredangle CBI$. We drag point $I$ until the two equations $\measuredangle BAI = \measuredangle CAI$ and $\measuredangle ABI = \measuredangle CBI$ hold simultaneously. The points $A, B$ and $C$ then are the required points. (See Figure 2)

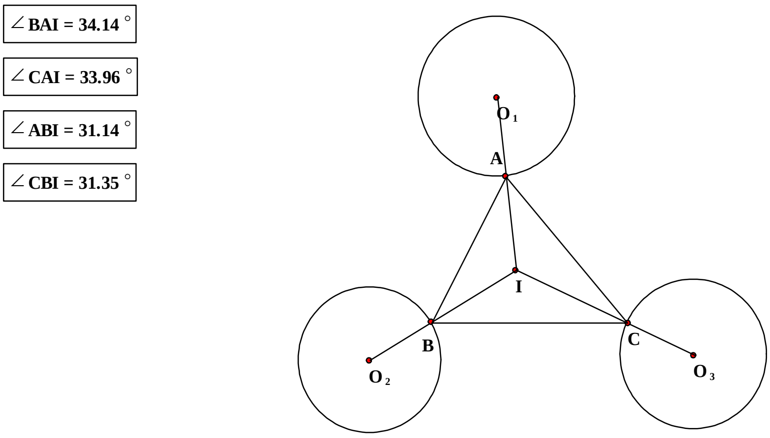


Figure 2. I is the incenter of $\triangle ABC$.

In Figure 2, we see that the angles are not identical due to restriction of package *[2]*. However, the points $A, B$ and $C$ are the best points we can obtain using *[2]*. (The error of deviation ratio is under 0.01%.)

The following is an immediate observation from Theorem 2:

**Theorem 3** *If $A \in \Gamma_1, B \in \Gamma_2$ and $C \in \Gamma_3$ such that the sum $AB + BC + CA$ reaches the minimum, and tangent lines for $\Gamma_1, \Gamma_2,$ and $\Gamma_3$ at $A, B,$ and $C$ respectively form a triangle $XYZ$. Then the point $I$, the incenter of triangle $ABC$, is the orthocenter of triangle $XYZ$, and $A, B, C$ are foots of the perpendiculars on the sides $YZ$, $ZX$ and $XY$.(see Figure 3)*

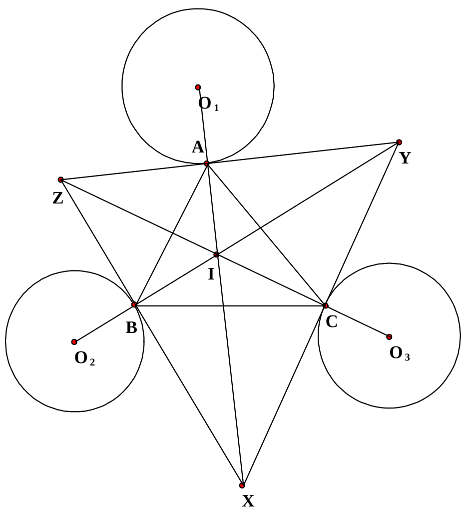


Figure 3. I is an incenter also an orthocenter

**Proof:** Let $\measuredangle BAZ = \measuredangle CAY = a$, $\measuredangle CBX = \measuredangle ABZ = b, \measuredangle ACY = \measuredangle BCX = c$, then $\measuredangle YXZ = \pi - b - c, \measuredangle ZYX = \pi - c - a$, and $\measuredangle XZY = \pi - a - b$. It follows from the fact that $\measuredangle YXZ +$ $\measuredangle ZYX + \measuredangle XZY = \pi$, we get $a + b + c = \pi$ and $\measuredangle YXZ = a, \measuredangle ZYX = b$ and $\measuredangle XZY = c$. Therefore, $A, B, X$ and $Y$ are cyclic, which implies that

$$\measuredangle XAY = \measuredangle XBY. \tag{3}$$

Similarly, it follows from $B, C, Y$ and $Z$ are cyclic, and $C, A, Z$ and $X$ are also cyclic that

$$\measuredangle YBZ = \measuredangle YCZ \text{ and} \tag{4}$$
$$\measuredangle ZCX = \measuredangle ZAX. \tag{5}$$

We know from equations (3)-(5) that each two of $\measuredangle XAY, \measuredangle YBZ$ and are $\measuredangle ZCX$ are supplementary angles, so all of them are right angles, which follow from the fact that $A, B$ and $C$ are being the feet of the perpendiculars on the sides $YZ, ZX$ and $XY$ respectively. Finally, it follows from $IA \perp YZ$, $IB \perp ZX$ and $IC \perp XY$ that $I$ is the orthocenter of triangle $XYZ$.

**Theorem 4** *The point $I$ satisfying $\measuredangle BAI = \measuredangle CAI$ and $\measuredangle ABI = \measuredangle CBI$ is unique.*

**Proof:** We prove by contradiction. If another point $I'$ also satisfies the given conditions, we let segments $I'O_1, I'O_2$ and $I'O_3$ meet $\Gamma_1, \Gamma_2$ and $\Gamma_3$ at points $A', B'$ and $C'$, respectively. Consider whether the points $A', B'$ and $C'$ are clockwise or counterclockwise of the points $A, B$ and $C$. From trigonometry Ceva's Theorem, we know that not all of them are clockwise, and not all of them are counterclockwise. Without loss of generality we suppose that point $A'$ is counterclockwise of point $A$, and points $B'$ and $C'$ are clockwise of points $B$ and $C$, respectively. (See Figure 4). We draw tangent line of $\Gamma_2$ and $\Gamma_3$ from points $B'$ and $C'$ respectively, and let these two tangent lines meet at point $X'$. It is clear that the points $O_1, A'$ and $X'$ are not collinear. We observe that the point $A'$ is on the right hand side of line $O_1A$, while the point $X'$ is on the left hand side line $O_1A$, which is a contradiction.

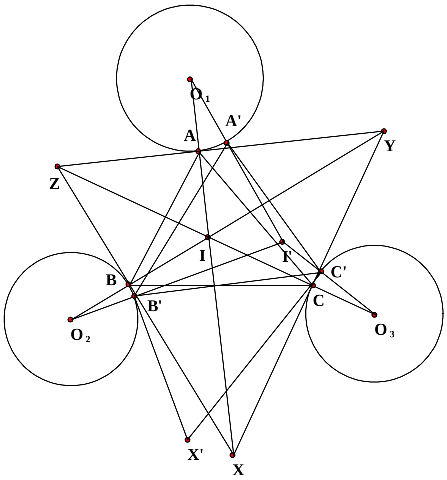


Figure 4. The uniqueness

We remark that students from China have access to a Dynamic Geometry System (DGS) *[2],* and they can use it to construct desired points described in Theorem 2. However, since most students do not have access to a Computer Algebra System (CAS), it is difficult for them to verify how accurate their answers are. We use the following example to show how we can find appropriate answers using Maple as follows:

**Example 1.** We are given three circles $C_4 : (x-2)^2 + (y-2)^2 - 1 = 0$, $C_5 : (x+3)^2 + (y+3)^2 - 4 = 0$, and $C_6 : (x-3)^2 + (y+2)^2 - 1/4 = 0$. (See Figure 5). We need to find appropriate points $A, B$ and $C$ on $C_4, C_5$ and $C_6$ respectively, so that the total distance of $AB + BC + CA$ achieves its minimum. Students were able to use geometric constructions with *[2]* to show that $AB + BC + CA$ achieves its minimum by using Theorem 2 with some error tolerance under 0.01%). However, we use the method described in Corollary and use Maple as the computational tool, we show that if we move $A'$ to $A = (1.76761632020814, 1.0273758046571)$, $B'$ to $B = (-1.20628618061282, -2.115358414871)$, and $C'$ to $C = (2.5843451027448, -1.72209532859668)$ accordingly as seen in Figure 5, $AB + BC + CA$ achieves its minimum. We note that the normal vectors at $A, B$ and $C$ respectively, pass through the **incenter** of triangle $ABC$.

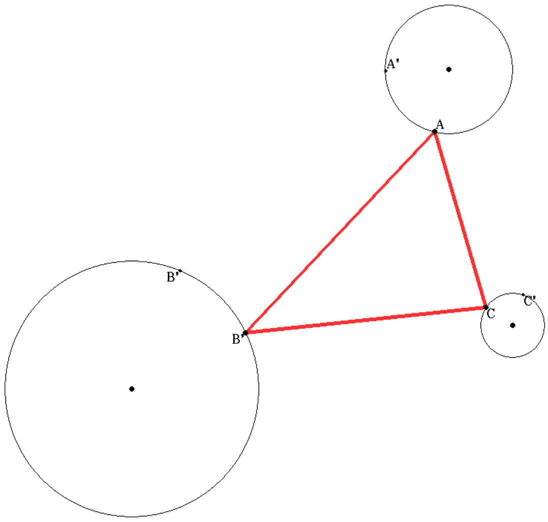


Figure 5. Shortest total distance for circles

**Remarks:** Through exploring Example 1, we conclude that for the circle case, if the points $A, B, C$ on $C_4, C_{5,}$ and $C_6$, respectively, are the desired points making the total distance $AB + BC + CA$ the shortest. Then the normal vectors at $A, B, C$ respectively should be the angle bisectors of $\measuredangle BAC, \measuredangle ABC$ and $\measuredangle BCA$ respectively and they should pass through the incenter of triangle $ABC$.We shall see how we generalize the circle case to a general case in 2D and corresponding ones in 3D by using Lagrange multiplier method below. We state the following Lagrange multiplier method (without proof), which can be found in many textbooks. We shall see that finding the minimum total distance is a special case of the Theorem 6.

**Theorem 5** *We assume that $f, g$ are continuously differentiable: $R^n \to R$. Suppose that we want to maximize or minimize a function of $n$ variables $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ for $\mathbf{x} = (x_1, x_2, ..., x_n)$ subject to $p$ constraints $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, ...,$ and $g_p(\mathbf{x}) = c_p$. The necessary condition for finding the relative maximum or minimum of $f(\mathbf{x})$ subject to the constraints $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, ...,$ and $g_p(\mathbf{x}) = c_p$ that is not on the boundary of the region where $f(\mathbf{x})$ and $g_i(\mathbf{x})$ are defined can be found by solving the system*

$$\frac{\partial}{\partial x_i}\left(f(\mathbf{x}) + \sum_{j=1}^{p} \lambda_j g_j(\mathbf{x})\right) = 0, \ 1 \le i \le n, \tag{6}$$

$$g_j(\mathbf{x}) = c_j, \ 1 \le j \le p. \tag{7}$$

*We write* $\nabla f(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} f(\mathbf{x}), \frac{\partial}{\partial x_2} f(\mathbf{x}), ..., \frac{\partial}{\partial x_n} f(\mathbf{x})\right)$. *If $x = \mathbf{x}_0$ is an extremum for the above system, then*

$$\nabla f(\mathbf{x}_0) = \sum_{j=1}^{p} \lambda_j \nabla g_j(\mathbf{x}_0). \tag{8}$$

Applying a similar approach to that taken in *[1]*, we remark that the Theorem 6 below is a generalization of Corollary. However, Corollary is inspired by finding shortest total distance function.

We assume that $f : R^{np} \to R, g_i : R^n \to R$, $i = 1, 2...p$, are continuously differentiable in their respective domains. Our objective is to maximize or minimize the function

$$\begin{aligned} f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) &= f(x_1^1, x_2^1, ..., x_n^1, x_1^2, x_2^2, ..., x_n^2, ...x_1^p, x_2^p, ..., x_n^p) \\ \text{for } \mathbf{x}_i &= (x_1^i, x_2^i, ..., x_n^i),\ i = 1, 2, ...p, \end{aligned} \tag{9}$$

subject to $p$ constraints

$$g_1(\mathbf{x}_1) = c_1, g_2(\mathbf{x}_2) = c_2, ..., \text{ and } g_p(\mathbf{x}_p) = c_p. \tag{10}$$

**Theorem 6** *The necessary condition of finding the relative extremum of $f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p)$ subject to the constraints $g_1(\mathbf{x}_1) = c_1, g_2(\mathbf{x}_2) = c_2, ...,$ and $g_p(\mathbf{x}_p) = c_p$ that is not on the boundary of the region where $f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p)$ and $g_i(\mathbf{x}_i)$ are defined can be found by solving the system*

$$\frac{\partial}{\partial \mathbf{x}_i}\left(f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) + \sum_{j=1}^{p} \lambda_j g_j(\mathbf{x}_j)\right) = 0,\ 1 \le i \le p, \tag{11}$$

$$g_j(\mathbf{x}_j) = c_j,\ 1 \le j \le p. \tag{12}$$

*If*

$$\sum_{i=1}^{p} \frac{\partial}{\partial \mathbf{x}_i}\left(f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p)\right) = 0, \tag{13}$$

*and $\mathbf{x} = \mathbf{x}_0 = (\mathbf{x}_1^*, \mathbf{x}_2^*, , , , \mathbf{x}_p^*)$ is an extremum for above system, then we have*

$$\sum_{j=1}^{p} \lambda_j \frac{\partial}{\partial \mathbf{x}_j}\left(g_j(\mathbf{x}_j^*)\right) = 0. \tag{14}$$

**Proof:** The proof can be found in *[1]*, theorem 4. For completeness, we include it here. The result follows directly from $\frac{\partial}{\partial \mathbf{x}_i} f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) = -\lambda_i \frac{\partial}{\partial \mathbf{x}_i} g_i(\mathbf{x}_i), i = 1, 2, ...p$. If $\sum_{i=1}^{p} \frac{\partial}{\partial \mathbf{x}_i}\left(f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p)\right) = 0$, and $\mathbf{x} = \mathbf{x}_0 = (\mathbf{x}_1^*, \mathbf{x}_2^*, , , , \mathbf{x}_p^*)$ is an extremum for above system, then

$$\sum_{j=1}^{p} \lambda_j \frac{\partial}{\partial \mathbf{x}_j}\left(g_j(\mathbf{x}_j^*)\right) = 0. \tag{15}$$

**Corollary.** *If the function*

$$\begin{aligned} f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) &= |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_2 - \mathbf{x}_3|^2 + ... + |\mathbf{x}_{p-1} - \mathbf{x}_p|^2 + |\mathbf{x}_p - \mathbf{x}_1|^2 \\ \text{for } \mathbf{x}_i &= (x_1^i, x_2^i, ..., x_n^i), \\ i &= 1, 2, ...p, \end{aligned} \tag{16}$$

*has an extremum subject to $p$ constraints*

$$g_1(\mathbf{x}_1) = c_1, g_2(\mathbf{x}_2) = c_2, ..., \text{ and } g_p(\mathbf{x}_p) = c_p, \tag{17}$$

*at* $\mathbf{x}_0 = (x_1^*, x_2^*, , , , x_p^*)$ *in its closed and bounded domain. Then we can find coefficients,* $\lambda_j, j = 1, 2, ...p$*, so that*

$$\sum_{j=1}^{p} \lambda_j \frac{\partial}{\partial \mathbf{x}_j} \left( g_j(\mathbf{x}_j^*) \right) = 0. \tag{18}$$

**Proof:** It is an easy exercise to see that $\sum_{i=1}^{p} \frac{\partial}{\partial \mathbf{x}_i} \left( f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) \right) = 0$ and the proof follows directly from above Theorem.

Now we consider the following generalized shortest total distance problem in $2D$, we will see that the answers obtained both from algebraic method of using Lagrange multipliers mentioned in Corollary and geometric construction using a DGS coincide with each other.
**Example 2.** Consider the following three curves in Figure 6(a) below, given by $C_1 : y = \sin(x)$ (shown in blue), $C_2 : y = x^2 + 2$ (a parabola shown in red), and $C_3 : (x-3)^2 + (y-3)^2 - 1 = 0$ (a circle shown in green). We need to find approximate points $A, B$ and $C$ on $C_1, C_2$ and $C_3$ respectively, so that the total distance of $AB + BC + CA$ achieves its minimum.

For simplicity, we use $g_i(\mathbf{x}_i) = 0$ to denote the equations for the curves of $C_i$, where $i = 1, 2,$and 3.We use $\mathbf{x}_i = (x_i, y_i), i = 1, 2$ and 3. Firstly, we solve this algebraically by consider the following equation:

$$L(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \lambda_1, \lambda_2, \lambda_3) = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{x}_2 - \mathbf{x}_3)^2 + (\mathbf{x}_3 - \mathbf{x}_1)^2} + \lambda_1 g_1(\mathbf{x}_1) + \lambda_2 g_2(\mathbf{x}_2) + \lambda_3 g_3(\mathbf{x}_3). \tag{19}$$

By setting the partial derivatives of $L$ equal to 0 and solving for the respective variables, we obtain the shortest total distance $AB + BC + CA$ to be 4.7490344570688468563 when

$$\begin{aligned} A &= (1.5361542615482507605, 0.99940002366259185632), \\ B &= (0.75230721377067956068, 2.5659661438914029544), \text{ and} \\ C &= (2.1500448920739965982, 2.4731448828088542621). \end{aligned} \tag{20}$$

Secondly, we see how our algebraic solution coincide with our geometric construction as we expected. We construct tangent lines at points at $C_1, C_2$ and $C_3$ respectively. We observe only when the points are at $A, B$ and $C$ mentioned in equation (20), will we achieve the minimum total shortest distance for $AB + BC + CA$. In such case, three normal lines at $A, B$ and $C$, respectively, meet at the incenter of $\triangle ABC$ (see Figure 6(b)).

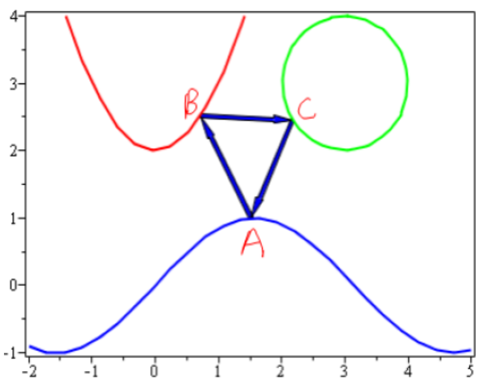


Figure 6(a) Total shortest distance obtained from a CAS

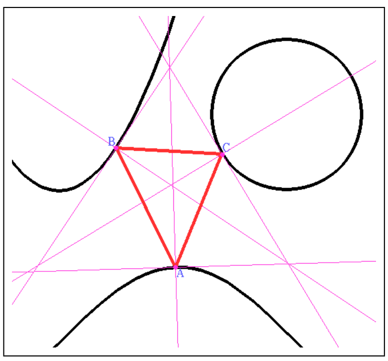

Figure 6(b) Total shortest distance obtained from a DGS

It is natural to generalize the $2D$ problem to the following $3D$ case:

**Example 3.** We are given four surfaces shown in Figure 7, where $S_1 : x^2 + y^2 + z^2 - 1 = 0$ (a sphere shown in yellow), $S_2 : x^2 + (y-3)^2 + (z-1)^2 - 1 = 0$ (a sphere shown in blue), $S_3 : z - (x^2 + y^2) - 2 = 0$ (a paraboloid shown in red), and $S_4 : (4(x-3) + (y-3) + (z-1))(x-3) + ((x-3) + 4(y-3) + (z-1))(y-3) + ((x-3) + (y-3) + 4(z-1))(z-1) - 3 = 0$ (an ellipsoid shown in green). We want to find points $A, B, C$ and $D$ on the surfaces $S_1, S_2, S_3$ and $S_4$ respectively, so that the total distance $AB + BC + CD + DA$ achieves its minimum.

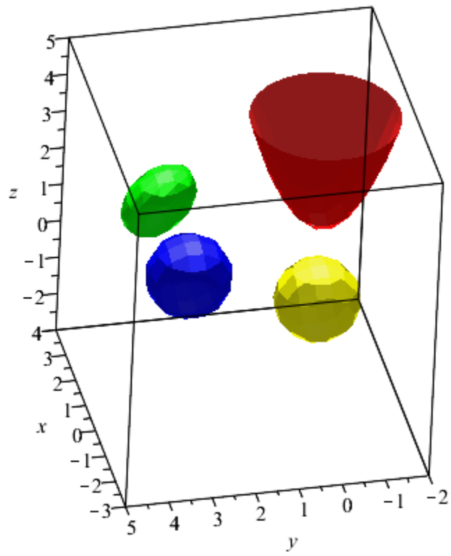


Figure 7. A total shortest distance problem in 3D

**Method 1: Geometric intuition:** We may think of this problem as a light beam reflection from one surface to the other. If we think of $BA$ as an incoming light beam toward the point $A$ on surface $S_1$ and $DA$ as an outgoing light beam at $A$, then $N_A$, the normal vector at $A$ on $S_1$ should be their angle bisector. Similarly, we should have the following observations:

1. The line segments $BC, AB$ and the normal vector at $B$, denoted by $N_B$, should lie on the same plane and $N_B$ is the angle bisector for $AB$ and $BC$.

2. The line segments $BC$, $CD$ and the normal vector at $C$, denoted by $N_C$, should lie on the same plane and $N_C$ is the angle bisector for $BC$ and $CD$.

3. The line segments $CD, DA$ and the normal vector at $D$, denoted by$N_D$, should lie on the same plane and $N_D$ is the angle bisector for $CD$ and $DA$.

We shall see if our geometric intuition is correct, but first we use Lagrange multiplier method stated in Theorem 6 or Corollary to find the solutions algebraically.

**Method 2: Solving it algebraically:** For simplicity, we use $g_i(\mathbf{x}_i) = 0$ to denote the equations for the surfaces of $S_i$, where $i = 1, 2, 3$, and 4. Moreover we use $\mathbf{x}_1 = (s_{11}, s_{12}, s_{13}) \in$

$S_1$, $\mathbf{x}_2 = (s_{21}, s_{22}, s_{23}) \in S_2, \mathbf{x}_3 = (s_{31}, s_{32}, s_{33}) \in S_3$, and $\mathbf{x}_4 = (s_{41}, s_{42}, s_{43}) \in S_4$ in our computations with Maple. We apply Lagrange Method by considering the following

$$L(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{x}_2 - \mathbf{x}_3)^2 + (\mathbf{x}_3 - \mathbf{x}_4)^2 + (\mathbf{x}_4 - \mathbf{x}_1)^2} + \lambda_1 g_1(\mathbf{x}_1) + \lambda_2 g_2(\mathbf{x}_2) + \lambda_3 g_3(\mathbf{x}_3) + \lambda_4 g_4(\mathbf{x}_4). \quad (21)$$

By setting the partial derivatives of $L = 0$ and solving for the variables with the help of Maple, we algebraically obtain the shortest total distance $AB + BC + CD + DA = 9.104983301$. The desired points $A, B, C$ and $D$ on $S_1, S_2, S_3$, and $S_4$, respectively, obtained from Maple are $A = [s_{11}, s_{12}, s_{13}], B = [s_{21}, s_{22}, s_{23}], C = [s_{31}, s_{32}, s_{33}]$ and $D = [s_{41}, s_{42}, s_{43}]$, where

$$s_{11} = .33389958127263692867, s_{12} = .82675923225542380177, s_{13} = 0.45274743677505224502;$$
$$s_{21} = 0.13641928282913798603, s_{22} = 2.0175068585849490438, s_{23} = 1.1268739782018690446;$$
$$s_{31} = 0.28488630072773836633, s_{32} = .62315636129160757197, s_{33} = 2.4694840549605319295;$$
$$s_{41} = 2.3607938428457871966, s_{42} = 2.4901551912577622423, s_{43} = 1.3195995907854243548.$$

The line segments of $AB, BC, CD$ and $DA$ are shown in Figure 8(a), and we also use Maple for computation to show in Figure 8(b) that the normal vector at $A$ is an angle bisector of $AB$ and $AD$. The angle $\alpha$ between $AB$ and $AD$ is 0.9679408165 and the angle $\beta$ between the normal vector at $A$ and $AB$ is equal to $\frac{\alpha}{2}$ with error of tolerance of $-1.5 \cdot 10^{-9}$.

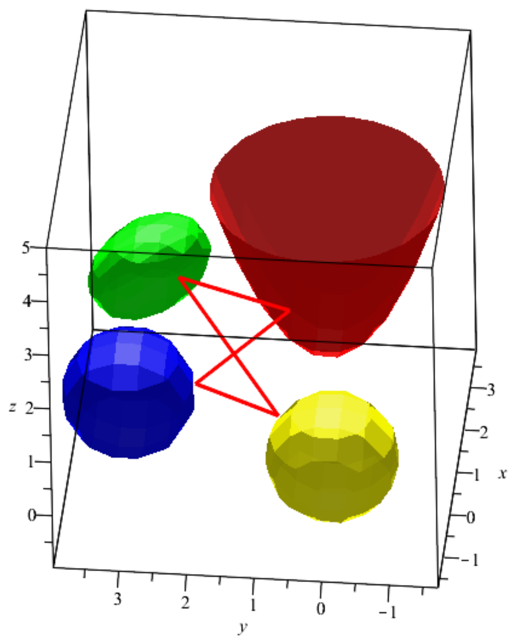


Figure 8(a). Shortest Total distance

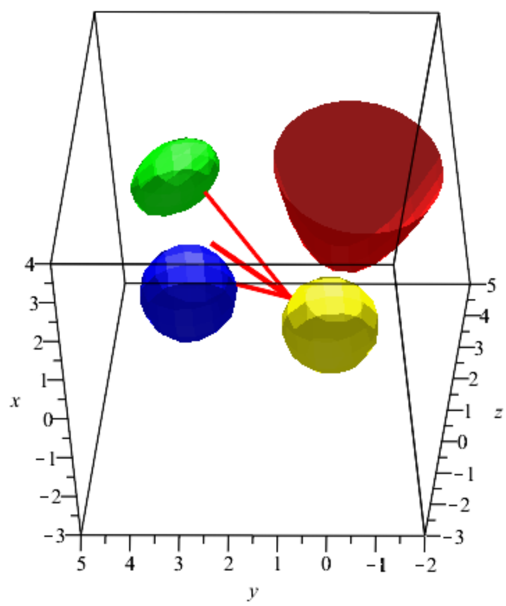


Figure 8(b) Normal vector is an angle bisector using Maple

Now, we shall again demonstrate this geometrically. The blue dotted lines in Figure 9 are respective normal vectors at respective points $A, B, C$ and $D$, and the dark red line segments are respective angle bisectors. In Figure 9, they coincide with respective normal vectors when the shortest total distance $AB + BC + CD + DA$ is achieved. In other words, **our geometry constructions coincide with our algebraic answers using CAS.**

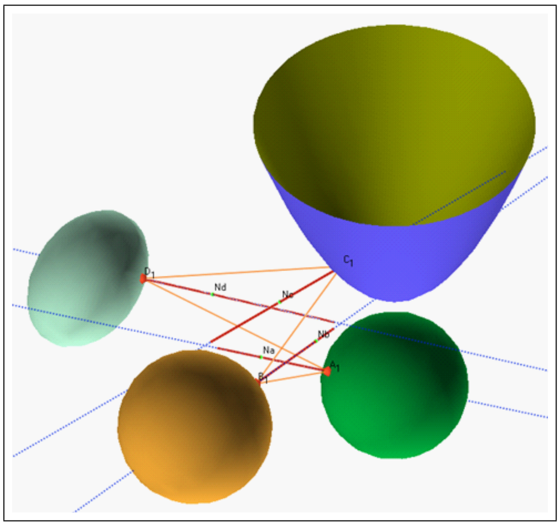

Figure 9. Total shortest distance 3D problem verified using a DGS

# 3 Conclusion

It is interesting to note the following observations for students with different backgrounds:

1. For students who have access to a CAS but not DGS: It is difficult for students to use geometric constructions to produce the simple intuitions. They can only modify the CAS worksheet when validating their algebraic answers using Lagrange multipliers method.

2. For students who have access to a DGS but no CAS: Students will use their favorite DGS to reproduce simple special cases (such as circles) discussed in this paper, and make conjecture about the validity of the solutions. However, their conjectures cannot be validated since they have no CAS to verify solutions analytically.

3. For those students who have no access to either a CAS or DGS, they can only appreciate the graphical representations of this paper, they have no available tools to experiment on their own.

It is natural to see how students first approach the optimization problem by considering the simplest case of circles in 2D, and use geometry constructions to gain intuition of what can be true in a general case in 2D and 3D. The Examples 2 and 3 indeed shed some lights on why we need to train students to equip the knowledge both in DGS and CAS when solving problems. In addition, we note that traditionally when technological tools are not available, students find applying Lagrange Multiplier Method in solving optimization problems difficult; not only due to the complexity of the algebraic manipulation nature but also students often do not fully understand the geometric interpretation behind the method. The problems described in this paper posed different ways of finding the shortest total distance compared to the ones given in *[1]*. Our aim is to emphasize the importance of DGS to gain geometric intuitions on the one hand, and also the necessity of being able to manipulate CAS on the other hand, to obtain solutions analytically in the areas of Linear Algebra and Multivariable Calculus.

# 4 Acknowledgements

Authors would like to thank Vladimir Nodelman for giving valuable inputs for this paper and using his own DGS to create Figure 9. The first author would also like to express his sincere gratitude and appreciation to following hosts for providing excellent research environments during his professional leave from Radford University betweem 2013 and 2014: Chinese Academy of Sciences in Beijing, National Institute of Education-Nanyang Technological University of Singapore, Guangzhou University of China, Beijing Normal University of China, and Center for Research on International Coorporation in Education Development-University of Tsukuba of Japan.

# References

[1] Yang, W.-C. *Some Geometric Interpretations of The Total Distances Among Curves and Surfaces*, the Electronic Journal of Mathematics and Technology (eJMT), Volume 3, Number 1, ISSN 1933-2823, Mathematics and Technology LLC.

[2] Z+Z Chinese Dynamic Geometry Software, http://www.worldscientific.com/doi/abs/10.1142/9789812776006_0007?queryID=%24%7BresultBean.queryID%7D&.